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UPPER BOUNDS ON RESPONSES OF LINEAR SYSTEMS UNDER TRANSIENT LOADS

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1. INTRODUCTION

The forced vibration of an n-degree-of-freedom linear second order system is represented by

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(t), \quad x(0) = \theta_n, \quad \dot{x}(0) = \theta_n \tag{1}$$

for all $t \geqslant 0$. In equation (1),

$$x(t) = [x_1(t) \quad x_2(t) \ \dots \ x_n(t)]^{\mathrm{T}} \in \mathbb{R}^n \tag{2}$$

denotes the vector of displacements (v^{T} denotes the transpose of a vector v); the mass matrix M, the damping matrix C, and the stiffness matrix K belong to $\mathbb{R}^{n \times n}$ and are symmetric and positive definite; $x(0) \in \mathbb{R}^n$ and $\dot{x}(0) \in \mathbb{R}^n$ are the vectors of initial displacements and velocities, respectively; θ_n denotes the zero vector in $\mathbb{R}^n$; the vector $f(t) \in \mathbb{R}^n$, $t \geqslant 0$, denotes the forcing vector. It is assumed that $f \in L_2^n(\mathbb{R}_+)$, that is, its norm, denoted by $\| f \|_2$, satisfies

$$\| f \|_2 := \left[\int_0^\infty f^{\mathrm{T}}(\tau) f(\tau)\, \mathrm{d}\tau \right]^{1/2} < \infty. \tag{3}$$

This assumption is readily satisfied for excitations encountered in practice, for instance, seismic and transient loadings.

In this note, the goal is to derive an *a priori* upper bound on the sizes (norms) of displacements of system (1) without solving it (numerically); more precisely, to derive a single upper bound on the L_∞-norm of the displacement $x_i(\cdot)$, defined by

$$\| x_i \|_\infty := \max_{t \geqslant 0} |x_i(t)| \tag{4}$$

for all $i = 1, 2, \dots, n$.

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In recent years, researchers have derived bounds on the sizes of displacements and velocities of free or forced vibratory systems; see, e.g., references [1; 2; 3, p. 136; 4–8; 9, pp. 177–178; 10]. Such bounds can be used in the design and analysis of systems. Bounds on the sizes of displacements of system (1) are useful when (1) they are easily computable; (2) they are tight. If the bounds are not easily computable, then one might as well solve system (1) (numerically) in order to obtain the exact (very accurate) values for the displacement peaks. If, on the other hand, the bounds are easily computable, but are conservatively large, then they furnish no useful information to be used in the system design and analysis. It appears that the two requirements of ease of computation and tightness of the upper bounds oppose each other: the less (more, respectively) computational effort, the more (less) conservative bounds on the sizes of displacements. Despite this fact, one should attempt to derive easy to compute and tight bounds; in particular, bounds on the norms of responses of forced systems, which are not easily computable and are usually conservative.

2. UPPER BOUNDS ON DISPLACEMENTS

In this section, two upper bounds on the norms of displacements of system (1) are obtained by two different approaches. In the following, $\lambda_{min}(H)$ and $\lambda_{max}(H)$ are used to denote the minimum and the maximum eigenvalues of a symmetric matrix H, respectively. A weighted norm for the forcing vector $f(\cdot)$ applied to system (1) is defined as

$$\| f \|_{2,H} := \left[\int_0^\infty f^{\mathrm{T}}(\tau) H f(\tau)\, \mathrm{d}\tau \right]^{1/2} \tag{5}$$

where H is a symmetric and positive-definite matrix. By the definition of Rayleigh's quotient (see, e.g., references [11, pp. 237–243; 12, pp. 176–181]) and inequality (3), it follows that $\| f \|_{2,H} \leqslant [\lambda_{max}(H)]^{1/2} \ \| f \|_2 < \infty$.

2.1. APPROACH 1

An upper bound is obtained by directly using system (1), which is the system representation in the physical co-ordinates.

Theorem 2.1. *Consider system* (1) *with* $f(\cdot)$ *satisfying inequality* (3). *The* L_∞*-norm of the displacement* $x_i(\cdot)$ *satisfies*

$$\|x_i\|_\infty \leqslant \frac{\| f \|_{2,C^{-1}}}{[2\,\lambda_{min}(K)]^{1/2}} \tag{6}$$

for all $i = 1, 2, \ldots, n$.

Proof. For system (1) consider the energy function

$$E(t) = \frac{1}{2}\dot{x}^{\mathrm{T}}(t)M\dot{x}(t) + \frac{1}{2}x^{\mathrm{T}}(t)Kx(t) \tag{7}$$

for all $t \geqslant 0$, where $E(0) = 0$. The derivative of $E(\cdot)$ along the solution of system (1) satisfies

$$\dot{E}(t) = -\dot{x}^{\mathrm{T}}(t)C\dot{x}(t) + \dot{x}^{\mathrm{T}}(t)f(t) \tag{8}$$

for all $t \geqslant 0$. Since the matrix C is positive definite, equation (8) can be written as

$$\dot{E}(t) = -\dot{x}^{\mathrm{T}}(t)C\dot{x}(t) + \dot{x}^{\mathrm{T}}(t)C^{1/2}C^{-1/2}f(t) \tag{9}$$

for all $t \geqslant 0$, where $C^{1/2}C^{1/2} = C$. By inequality (A1), established in Appendix A,

$$\dot{x}^{\mathrm{T}}(t)C^{1/2}\ C^{-1/2}f(t) \leqslant \dot{x}^{\mathrm{T}}(t)C\dot{x}(t) + \frac{1}{4}f^{\mathrm{T}}(t)C^{-1}f(t) \tag{10}$$

for all $t \geqslant 0$. Using inequality (10) in equation (9), it is concluded that

$$\dot{E}(t) \leqslant \frac{1}{4}f^{\mathrm{T}}(t)C^{-1}f(t) \tag{11}$$

for all $t \geqslant 0$. Having $E(0) = 0$ and using a theorem from the theory of differential inequalities (see, e.g., references [13, p. 2; 14, p. 3]), it is concluded that $E(\cdot)$ in inequality (11) satisfies

$$E(t) \leqslant \frac{1}{4}\int_0^t f^{\mathrm{T}}(\tau)\,C^{-1}f(\tau)\,\mathrm{d}\tau \leqslant \frac{\|f\|^2_{2,C^{-1}}}{4} \tag{12}$$

for all $t \geqslant 0$. Thus, it follows from inequality (12) that

$$x^{\mathrm{T}}(t)Kx(t)/2 \leqslant \|f\|^2_{2,C^{-1}}/4 \tag{13}$$

for all $t \geqslant 0$. Using the definition of Rayleigh's quotient in the left-hand side of inequality (13), it is concluded that

$$\lambda_{min}(K)x^{\mathrm{T}}(t)x(t) \leqslant \|f\|^2_{2,C^{-1}}/2 \tag{14}$$

for all $t \geqslant 0$. Finally, using $x^{\mathrm{T}}(\cdot)x(\cdot)$ from inequality (14) in

$$\|x_i\|_\infty = \max_{t \geqslant 0} |x_i(t)| \leqslant \max_{t \geqslant 0} [x^{\mathrm{T}}(t)x(t)]^{1/2} \tag{15}$$

the upper bound in inequality (6) is established. $\square$

Next, a different upper bound is obtained,

2.2. APPROACH 2

An upper bound is obtained by using the following representation of system (1):

$$MK^{-1}M\ddot{x}(t) + MK^{-1}C\dot{x}(t) + Mx(t) = MK^{-1}f(t), \quad x(0) = \theta_n, \quad \dot{x}(0) = \theta_n \tag{16}$$

for all $t \geqslant 0$.

Theorem 2.2. *Consider system* (1) *with* $f(\cdot)$ *satisfying inequality* (3). *Let the matrix* $S := CM^{-1}K + KM^{-1}C$ *be positive definite. The* L_∞*-norm of the displacement* $x_i(\cdot)$ *satisfies*

$$\|x_i\|_\infty \leqslant \frac{\|f\|_{2,S^{-1}}}{[\lambda_{min}(M)]^{1/2}} \tag{17}$$

for all $i = 1, 2, \ldots, n$.

Proof. For system (16) consider the energy function

$$V(t) = \frac{1}{2}\dot{x}^{\mathrm{T}}(t)MK^{-1}M\dot{x}(t) + \frac{1}{2}x^{\mathrm{T}}(t)Mx(t) \tag{18}$$

for all $t \geqslant 0$, where $V(0) = 0$. The derivative of $V(\cdot)$ along the solution of system (16) satisfies

$$\dot{V}(t) = -\dot{x}^{\mathrm{T}}(t)MK^{-1}C\dot{x}(t) + \dot{x}^{\mathrm{T}}(t)MK^{-1}f(t) \tag{19}$$

for all $t \geqslant 0$. Since $MK^{-1}C$ is not a symmetric matrix, its asymmetric part vanishes in equation (19). Thus,

$$\dot{V}(t) = -\frac{1}{2}\dot{x}^{\mathrm{T}}(t)(MK^{-1}C + CK^{-1}M)\dot{x}(t) + \dot{x}^{\mathrm{T}}(t)MK^{-1}f(t) \tag{20}$$

for all $t \geqslant 0$. Clearly,

$$MK^{-1}C + CK^{-1}M = MK^{-1}(CM^{-1}K + KM^{-1}C)K^{-1}M. \tag{21}$$

From equation (21), it is concluded that the matrix $MK^{-1}C + CK^{-1}M$ is positive definite due to the fact that $S = CM^{-1}K + KM^{-1}C$ is a positive-definite matrix and M and K are non-singular matrices. Thus, equation (20) can be written as

$$\begin{aligned}\dot{V}(t) = &-\frac{1}{2}\dot{x}^{\mathrm{T}}(t)(MK^{-1}C + CK^{-1}M)\dot{x}(t)\\ &+\dot{x}^{\mathrm{T}}(t)[(MK^{-1}C + CK^{-1}M)/2]^{1/2}\\ &\times[(MK^{-1}C + CK^{-1}M)/2]^{-1/2}MK^{-1}f(t)\end{aligned} \tag{22}$$

for all $t \geqslant 0$. Applying inequality (A1) in Appendix A to the second term on the right-hand side of equation (22), it is concluded that

$$\dot{V}(t) \leqslant \frac{1}{2} f^{\mathrm{T}}(t)K^{-1}M(MK^{-1}C + CK^{-1}M)^{-1}MK^{-1}f(t) \tag{23}$$

for all $t \geqslant 0$. Using equation (21) in inequality (23), it is concluded that

$$\dot{V}(t) \leqslant \frac{1}{2} f^{\mathrm{T}}(t)(CM^{-1}K + KM^{-1}C)^{-1}f(t) = \frac{1}{2}f^{\mathrm{T}}(t)S^{-1}f(t) \tag{24}$$

for all $t \geqslant 0$. Having $V(0) = 0$ and using a theorem from the theory of differential inequalities, it is concluded that $V(\cdot)$ in inequality (24) satisfies

$$V(t) \leqslant \frac{1}{2}\int_0^t f^{\mathrm{T}}(\tau)S^{-1}f(\tau)\,\mathrm{d}\tau \leqslant \frac{\|f\|^2_{2,S^{-1}}}{2} \tag{25}$$

for all $t \geqslant 0$. Thus, it follows from inequality (25) that

$$x^{\mathrm{T}}(t)Mx(t)/2 \leqslant \|f\|^2_{2,S^{-1}}/2 \tag{26}$$

for all $t \geqslant 0$. Using the definition of Rayleigh's quotient on the left-hand side of inequality (26), it is concluded that

$$\lambda_{min}(M)\,x^{\mathrm{T}}(t)x(t) \leqslant \|f\|^2_{2,S^{-1}} \tag{27}$$

for all $t \geqslant 0$. Finally, following the last step in the proof of Theorem 2.1, the upper bound in inequality (17) is established. □

Remark. In Theorem 2.2, the assumption that $S = CM^{-1}K + KM^{-1}C$ is a positive-definite matrix is not a restrictive assumption. The reason is as follows. In reference [8], it is shown that if system (1) is classically damped, i.e., if $CM^{-1}K = KM^{-1}C$, then the matrix S is positive definite. However, it is further shown that when S is positive definite, it is not necessarily true that

$CM^{-1}K = KM^{-1}C$. Thus, the class of classically damped systems is a subclass of the systems for which S is positive definite. It happens that for systems encountered in practise, the matrix S is usually positive definite, where as the identity $CM^{-1}K = KM^{-1}C$ rarely holds. □

3. EXAMPLE

In this section, an example is given to study the bounds obtained in this note. Consider the system in Figure 1 and let $m_i = 1$ and $k_i = 10$ for all $i = 1, 2, 3$, and $c_1 = 0{\cdot}66$, $c_2 = 0{\cdot}6$, $c_3 = 0{\cdot}66$.

Furthermore, let the system be excited by the force

$$f(t) = \begin{cases} [0{\cdot}2 \;\; 0{\cdot}2 \;\; 0{\cdot}2]^{\mathrm{T}} \sin \dfrac{9\pi}{20} t, & 0 \leqslant t \leqslant 20, \\ \theta_3, & 20 < t. \end{cases} \tag{28}$$

The vibration of this system is represented by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \ddot{x}_3(t) \end{bmatrix} + \begin{bmatrix} 1{\cdot}26 & -0{\cdot}6 & 0 \\ -0{\cdot}6 & 1{\cdot}26 & -0{\cdot}66 \\ 0 & -0{\cdot}66 & 0{\cdot}66 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 20 & -10 & 0 \\ -10 & 20 & -10 \\ 0 & -10 & 10 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0{\cdot}2 \\ 0{\cdot}2 \\ 0{\cdot}2 \end{bmatrix} \sin \frac{9\pi}{20} t \tag{29}$$

for all $0 \leqslant t \leqslant 20$, where the right-hand side of equation (29) is zero for all $t > 20$. The vectors of initial displacements and velocities of system (29) are zero.

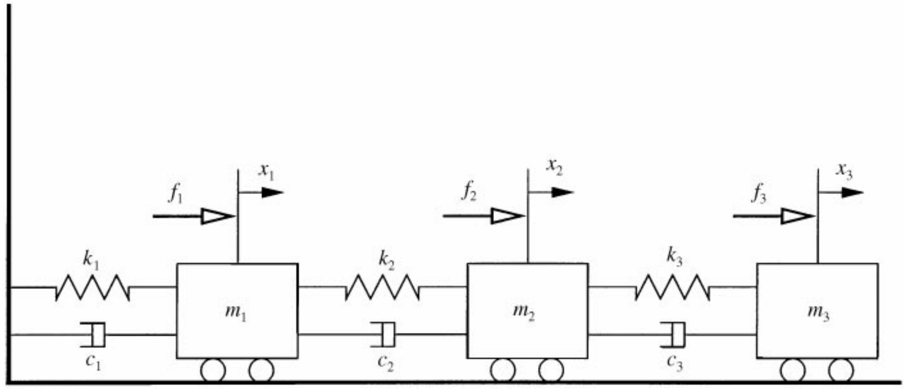

Figure 1. The system represented by equation (29) is a three-degrees-of-freedom system excited by forces $f_1(\cdot)$, $f_2(\cdot)$, and $f_3(\cdot)$.

Having the coefficient matrices M, C, and K of system (29) identified, by straightforward computation, it is obtained that

$$\| f \|_{2,C^{-1}} = \left[\int_0^{20} f^{\mathrm{T}}(\tau) C^{-1} f(\tau)\, \mathrm{d}\tau \right]^{1/2} = \left[0{\cdot}8727 \int_0^{20} \sin^2 \frac{9\pi}{20} \tau \, \mathrm{d}\tau \right]^{1/2} = 2{\cdot}9542 \tag{30a}$$

$$\lambda_{min}(K) = 1{\cdot}9806. \tag{30b}$$

Therefore, by inequality (6), it follows that

$$\| x_1 \|_\infty \leqslant 1{\cdot}4843, \qquad \| x_2 \|_\infty \leqslant 1{\cdot}4843, \qquad \| x_3 \|_\infty \leqslant 1{\cdot}4843. \tag{31}$$

It is next verified that the matrix $S = CM^{-1}K + KM^{-1}C$ is positive definite. Thus, the upper bound in inequality (17) can be computed. By straightforward computation, it is obtained that

$$\| f \|_{2,S^{-1}} = 2{\cdot}2026. \tag{32}$$

Therefore, by inequality (17), it follows that

$$\| x_1 \|_\infty \leqslant 1{\cdot}4841, \qquad \| x_2 \|_\infty \leqslant 1{\cdot}4841, \qquad \| x_3 \|_\infty \leqslant 1{\cdot}4841. \tag{33}$$

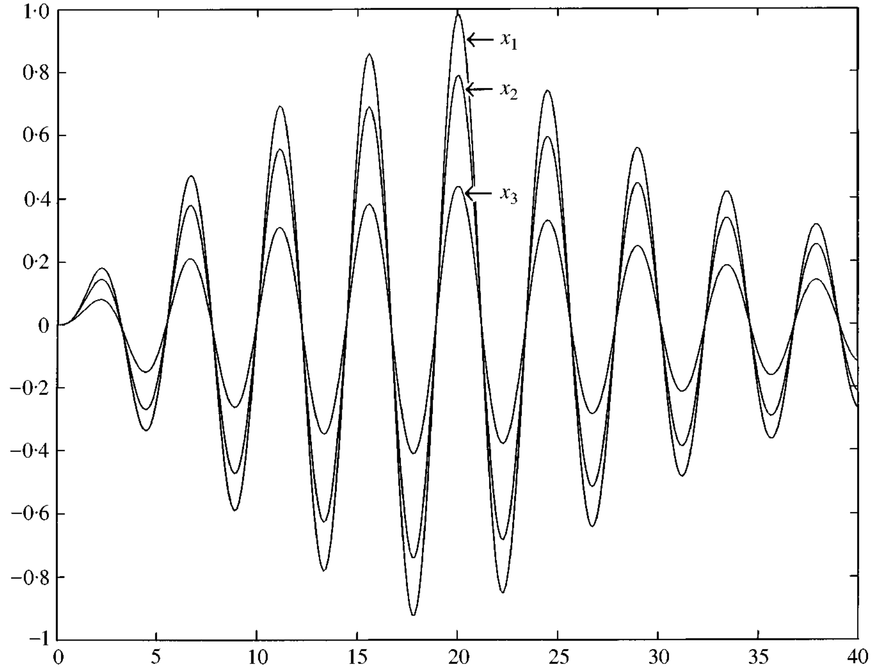

Figure 2. Responses of system (29).

Responses of system (29) are obtained by the numerical integration and are plotted in Figure 2. From this figure, it follows that

$$\|x_1\|_\infty = 0{\cdot}9814, \qquad \|x_2\|_\infty = 0{\cdot}7870, \qquad \|x_3\|_\infty = 0{\cdot}4366. \tag{34}$$

Obtaining upper bounds on the norms of responses of forced systems is usually a difficult task. Moreover, most of the available bounds are conservative. Comparing the exact value of $\|x_1\|_\infty$ in equation (34) and the upper bounds corresponding to it in inequalities (31) and (33), it is concluded that the bounds are not very conservative. The upper bounds in inequalities (31) and (33) are close to each other; however, this is not always the case. There can be systems for which the bound obtained by inequality (6) is much different from that obtained by inequality (17).

4. CONCLUSIONS

In this note, the forced vibration of n-degree-of-freedom linear second order systems is considered. Two easy-to-compute upper bounds on the norms of displacements of such systems are derived. The upper bounds are given by inequalities (6) and (17). Each bound is a single upper bound on the norms of all displacements of system (1) and hence is computed only once.

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APPENDIX A: USEFUL INEQUALITY

The following useful inequality, although is available in the literature, is established here for completeness.

Lemma A.1. *For the vectors v_1 and v_2 in $\mathbb{R}^n$, the following inequality holds*:

$$v_1^{\mathrm{T}} v_2 \leqslant v_1^{\mathrm{T}} v_1 + v_2^{\mathrm{T}} v_2/4. \tag{A1}$$

Proof. Clearly,

$$(v_1 - v_2/2)^{\mathrm{T}} (v_1 - v_2/2) \geqslant 0. \tag{A2}$$

By expanding the right-hand side of inequality (A2), inequality (A1) follows. □